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An extended phase-space SUSY quantum mechanics

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Abstract

In the present paper, we will concern ourselves with the *extended* phase-space quantum mechanics of particles which have both bosonic and fermionic degrees of freedom, i.e., the quantum field theory in $(0 + 1)$ dimensions in q-(position) and p-(momentum) spaces, exhibiting supersymmetry. We present $(N = 2)$ realization of *extended* supersymmetry algebra and discuss the vacuum energy and topology of super-potentials. Shape invariance of exactly solvable extended SUSY potentials allows us to obtain analytic expressions for the entire energy spectrum of an extended Hamiltonian with, for example, *Scarf* potential without ever referring to an underlying differential equation.

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1. Introduction

From its historical development, the phase-space quantization is constructed on the premises that p and q are independent variables. In reducing the theory to that of Schrödinger or Heisenberg, the standard ordering emerges as the rule of game. However, if one conjectures to keep the symmetry between canonical coordinates and momenta in the process of quantization, at once, one may arrive at state functions in a phase-space representation. But this aspect of statistical quantum mechanics which deserves further investigation, unfortunately, has attracted little attention in subsequent developments.

The purpose of the papers [1, 2] is to fill this gap and to illustrate the usefulness of this perspective. In [1], it was observed that the concept of an *extended* Lagrangian, $\mathcal{L}(p, q, \dot{p}, \dot{q})$ in phase space allows a subsequent extension of Hamilton's principle to actions minimum along the actual trajectories in (p, q), rather than in q space. This extension, in turn, allows a definition of 'second' momenta $\pi_p = \delta\mathcal{L}/\delta\dot{p}$ and $\pi_q = \delta\mathcal{L}/\delta\dot{q}$, and a subsequent introduction of an *extended* phase space (p, q, π_p, π_q) and of an *extended* Hamiltonian, $H_{\text{ext}}(p, q, \pi_p, \pi_q)$. In particular, vanishing of π_q or π_p is the condition for p and q to constitute a canonical pair. In the language of statistical quantum mechanics, this choice picks up a pure state (*actual path*). Otherwise, one is dealing with a mixed state (*virtual path*).

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This simple formalism manifests its practical and technical virtue in the proposed canonical quantization in (p, q) space. It was first of all the unifying aspect that at once provides a framework for quantum statistical mechanics, for the classical statistical mechanics (Liouville's equation), for the conventional quantum mechanics as a special case, for von Neumann's density matrix and its equation of evolution as its inevitable corollaries. Wigner's [3] distributions and the equation satisfied by them are also obtained from those of [1] by an appropriate canonical transformation in the proposed (p, q, π_p, π_q) space. Further conceptual and practical merits of the formalism are demonstrated by treatment of Bloch's equation, partition functions for simple harmonic and linear potentials etc.

There is another line of reasoning which supports the side of *extended* phase-space formulation. In [2], we addressed the question of extended phase-space stochastic quantization of Hamiltonian systems with first class holonomic constraints and have proved that Lagrange's method of indeterminate multipliers yields the quantization of constrained systems in the stochastic quantization method. This in a natural way results in the Faddeev–Popov conventional path-integral measure for gauge systems.

On the other hand, for more than four decades, the inspiring idea of supersymmetry (SUSY) [4, 5] has led to new insights in a quantum field theory which unifies bosons and fermions, in particular SUSY quantum mechanics [6–10]. All this variety prompts us in the present paper to continue this program towards the supersymmetrization and, thus, to address the extended phase-space quantum mechanics of particles with odd degrees of freedom. This allows us to amplify and substantiate the assertions made in [1, 2].

This paper has been organized as follows. In the first part (sections 2 and 3), we give the appropriate definition of the extended phase-space SUSY quantum mechanical system and show how to construct the $(N = 2)$-SUSY algebra. In the second part (sections 4 and 5), we explore the vacuum energy and topology of super-potentials, and deal with the shape invariance of exactly solvable SUSY potentials. As an application, we obtain analytic expressions for the entire energy spectrum of extended Hamiltonian with Scarf potential without ever referring to the underlying differential equation. An implicit summation on repeated indices and the units $(\hbar = c = 1)$ are assumed throughout this paper.

2. SUSY in the extended phase-space quantum mechanical system

Consider a dynamical system with N degrees of freedom described by the $2N$ coordinates $q = (q_1, \ldots, q_N)$ and momenta $p = (p_1, \ldots, p_N)$ and a Lagrangian $\mathcal{L}^q(q, \dot{q})$ in q representation and the corresponding $\mathcal{L}^p(p, \dot{p})$ in p representation. In the framework of the proposed extended phase-space formalism of [1], the extended Lagrangian is written as

$$\mathcal{L}_{\text{ext}}(p, q, \dot{p}, \dot{q}) = -\dot{q}_i p_i - q_i \dot{p}_i + \mathcal{L}^q + \mathcal{L}^p, \tag{1}$$

where p and q are independent and not, in general, canonical pairs. A dot will indicate differentiation with respect to t. The independent nature of p and q gives the freedom of introducing a second set of canonical momenta for both p and q through the extended Lagrangian

$$\pi_{q_i} = \frac{\partial \mathcal{L}_{\text{ext}}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}_i} - p_i, \qquad \pi_{p_i} = \frac{\partial \mathcal{L}_{\text{ext}}}{\partial \dot{p}_i} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}_i} - q_i.$$

One may now define an extended Hamiltonian

$$H_{\text{ext}}(p, q, \pi_p, \pi_q) = \pi_{q_i}\dot{q}_i + \pi_{p_i}\dot{p}_i - \mathcal{L}_{\text{ext}}(p, q, \dot{p}, \dot{q}) = H(p + \pi_q, q) - H(p, q + \pi_p), \tag{2}$$

where $H(p, q) = p_i\dot{q}_i - \mathcal{L}^q = q_i\dot{p}_i - \mathcal{L}^p$ is the conventional Hamiltonian of the system. Here, p and q will be considered as independent c-number operators on the integrable complex function $\chi(q, p)$. For π_p and π_q, however, the differential operators and commutation brackets will be borrowed from the conventional quantum mechanics

$$\pi_{q_i} = -\mathrm{i}\frac{\partial}{\partial q_i}, \qquad [\pi_{q_i}, q_j] = -\mathrm{i}\delta_{ij}, \qquad \pi_{p_i} = -\mathrm{i}\frac{\partial}{\partial p_i}, \qquad [\pi_{p_i}, p_j] = -\mathrm{i}\delta_{ij}. \tag{3}$$

Note also the following

$$[p_i, q_j] = [p_i, p_j] = [q_i, q_j] = [\pi_{p_i}, \pi_{q_j}] = [\pi_{p_i}, \pi_{p_j}] = [\pi_{q_i}, \pi_{q_j}] = 0. \tag{4}$$

By the virtue of equations (3) and (4), H_{ext} is now an operator on χ. Along the trajectories in (p, q) space, however, it produces the state functions, $\chi(p, q, t)$, via the following Schrödinger-like equation:

$$\mathrm{i}\frac{\partial}{\partial t}\chi = H_{\mathrm{ext}}\chi. \tag{5}$$

Solutions of equation (5) are

$$\chi(q, p, t) = a_{\alpha\beta}\psi_\alpha(q, t)\phi^*_\beta(p, t)\,\mathrm{e}^{-\mathrm{i}pq}, \tag{6}$$

where $a = a^\dagger$, positive definite, $\mathrm{tr}\, a = 1$ and ψ_α and ϕ^*_α are the solutions of the conventional Schrödinger equation in q and p representations, respectively. The normalized χ is a physically acceptable solution. See [1] for further details.

Following a general prescription of SUSY quantum mechanics [6, 7], we call an extended phase-space quantum mechanical system characterized by an extended Hamiltonian H_{ext} acting in some Hilbert space $\mathcal{H}$ supersymmetric if there exist self-adjoint operators $Q_i = Q_i^\dagger, i = 1, 2, \ldots, N$, called *supercharges*, which also act on states in $\mathcal{H}$ and fulfill the following SUSY algebra:

$$\begin{aligned} &\{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i = 2H_{\mathrm{ext}}\delta_{ij}, \\ &[Q_i, H_{\mathrm{ext}}] = Q_iH_{\mathrm{ext}} - H_{\mathrm{ext}}Q_i = 0, \qquad i, j = 1, \ldots, N. \end{aligned} \tag{7}$$

Pursuing the analogy with these ideas in outlined here approach let a self-adjoint operator $P = P^\dagger$ be *Witten operator* or *Witten parity*, which anticommutes with the supercharges, and therefore commutes with an extended Hamiltonian, and whose square is equal to the identity [8]

$$\{Q_i, P\} = 0, \qquad [H_{\mathrm{ext}}, P] = 0, \qquad P^2 = 1. \tag{8}$$

This operator allows us to introduce the notion of *bosonic* and *fermionic* states independently of an underlying space-time symmetry. The Witten parity can also be written in the form $P = (-1)^{n_F}$ where n_F is the fermion-number operator. Therefore, eigenstates of P with eigenvalue -1 correspond to *fermions* and those with +1 correspond to *bosons*. In accordance, the bosonic $\mathcal{H}_B$- and fermionic $\mathcal{H}_B$- subspaces read

$$\mathcal{H}_B = \{\chi \in \mathcal{H} | P\chi >= +\chi\}, \qquad \mathcal{H}_F = \{\chi \in \mathcal{H} | P\chi >= -\chi\}. \tag{9}$$

Hence, any state $\chi \in \mathcal{H}$ can be decomposed into its bosonic and fermionic components. The Hilbert space may be written as a product space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{C}^2$, and thus, the Witten operator is represented by the third Pauli matrix σ_3:

$$P = \mathbf{1} \otimes \sigma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{10}$$

It will be more appropriate to use the notion spin-up and spin-down states (of a fictitious spin-$\frac{1}{2}$ particle with mass $m > 0$ moving along the d-dimensional Euclidean line $\mathcal{R}^d$) instead of bosonic and fermionic states, respectively. Having in addition only Cartesian degrees of freedom $\mathcal{H}_0$ is given by the space of square-integrable functions over the

$\mathcal{R}^d, \mathcal{H}_0 = \mathcal{L}^2(\mathcal{R}^d) \otimes \mathcal{L}^2(\mathcal{R}^d), d \in \mathbf{N}$. The SUSY has also implications on the spectral properties of the extended Hamiltonian H_{ext}. First of all, we note $H_{\text{ext}} = Q_i^2 \geqslant 0$, that is, the extended Hamiltonian has only non-negative eigenvalues. Suppose that χ_r is an eigenstate of H_{ext} with a positive eigenvalue $E_r > 0$. Then it follows immediately from the algebra equation (7) that

$$\widetilde{\chi}_r(q, p) = \frac{1}{\sqrt{E_r}} Q_i \chi_r(q, p), \qquad i = 1, 2 \ldots, N, \tag{11}$$

is also an eigenstate with the same positive eigenvalue. Hence, all positive-energy eigenstates occur in spin-up (boson) spin-down (fermion) pairs. Actually, a multiplicity of degeneracy of the levels of Hamiltonian H_{ext} with the energy E equals to a dimension of invariant subspace with respect to the action of all the Q_i. If $E = 0$, then the corresponding subspace is one dimensional—a level of zero point energy.

In general, the superalgebra equation (7) defines the Clifford algebra with the basis of $q_i = \frac{Q_i}{\sqrt{E}}$ for nonzero-energy levels of H_{ext}, which is a key point in the SUSY theories. Due to it a definition of the multiplicity of degeneracy of the energy levels reduced to a definition of a dimension of the representations of the Clifford algebra, which is well known. For the even and odd number N, a dimension of the representation of Clifford algebra is given as $\nu = 2^n = 2^{[N/2]}$, where $[\ldots]$ means the integer part, namely the ν defines a number of states in given supermultiplet.

3. The (N = 2)-SUSY in extended phase space

In constructing a particular ($N = 2$) realization of the SUSY algebra equation (7) in the Hilbert space

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{C}^2 = [\mathcal{L}^2(\mathcal{R}) \otimes \mathcal{L}^2(\mathcal{R})] \otimes \mathcal{C}^2, \tag{12}$$

following [8] let us first introduce a bosonic operator $B_\pm$ in q and p representations and a fermionic operator $\hat{f}$:

$$\begin{aligned} &B_{q\pm} : \mathcal{L}^2(\mathcal{R}) \to \mathcal{L}^2(\mathcal{R}), \qquad B_{q\pm} = [p + \pi_q \pm \mathrm{i}W(q)], \\ &B_{p\pm} : \mathcal{L}^2(\mathcal{R}) \to \mathcal{L}^2(\mathcal{R}), \qquad B_{p\pm} = [q + \pi_p \pm \mathrm{i}V(p)], \end{aligned} \tag{13}$$

and

$$\hat{f} : \mathcal{C}^2 \to \mathcal{C}^2, \qquad \hat{f} = \tfrac{1}{2}[\hat{\psi}_+, \hat{\psi}_-]. \tag{14}$$

As in canonical quantum mechanics, in expressions (13) and (14), the observables (π_q, q) and (π_p, p) are usual bosonic momentum and coordinate operators respectively in q and p spaces, while $\hat{\psi}_\pm$ are two real fermionic creation and annihilation nilpotent operators describing the fermionic variables, $W(q) : \mathcal{R} \to \mathcal{R}$ and $V(p) : \mathcal{R} \to \mathcal{R}$ are the piecewise continuously differentiable functions called SUSY potentials. The bosonic operators can, in the (q, p) representation, be taken in the usual form of equation (3). The $\hat{\psi}_\pm$, having anticommuting c-number eigenvalues, imply

$$\hat{\psi}_\pm = \sqrt{\tfrac{1}{2}}(\hat{\psi}_1 \pm \mathrm{i}\hat{\psi}_2), \qquad \{\hat{\psi}_\alpha, \hat{\psi}_\beta\} = \delta_{\alpha\beta}, \qquad \{\hat{\psi}_+, \hat{\psi}_-\} = 1, \qquad \hat{\psi}_\pm^2 = 0. \tag{15}$$

They can be represented by finite dimensional matrices $\hat{\psi}_\pm = \sigma^\pm$,

$$\hat{\psi}_+ = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\psi}_- = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{f} = \frac{1}{2}\sigma_3, \tag{16}$$

where $\sigma^\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$ denote the usual raising and lowering operators for the eigenvalues of σ_3. The fermionic operator, equation (14), commutes with the H_{ext} and is diagonal in this representation with conserved eigenvalues $\pm\frac{1}{2}$. Due to it, the wavefunctions become two-component objects:

$$\chi(q,p) = \begin{pmatrix} \chi_{+1/2}(q,p) \\ \chi_{-1/2}(q,p) \end{pmatrix} = \begin{pmatrix} \chi_1(q,p) \\ \chi_2(q,p) \end{pmatrix} = \begin{pmatrix} \psi_1(q)\phi_1(p) \\ \psi_2(q)\phi_2(p) \end{pmatrix}, \tag{17}$$

where the states $\psi_{1,2}(q), \phi_{1,2}(p)$ correspond to the fermionic quantum number $f = \pm\frac{1}{2}$, respectively, in q and p spaces.

Let us now to deal with an abstract space of the eigenstates of the conjugate operator $\hat{\psi}_\pm$ having anticommuting c-number eigenvalues. Suppose $|00-\rangle$ is the normalized zero-eigenstate of $\hat{q}$ and $\hat{\psi}_-$:

$$\hat{q}|00-\rangle = 0, \qquad \hat{\psi}_-|00-\rangle = 0. \tag{18}$$

The state $|00+\rangle$ can be defined by

$$|00+\rangle = \hat{\psi}_+|00-\rangle, \tag{19}$$

then

$$\hat{\psi}_+|00+\rangle = 0, \qquad \hat{\psi}_-|00+\rangle = |00-\rangle. \tag{20}$$

Taking into account that $\hat{\psi}_\pm^\dagger = \hat{\psi}_\mp$, we get

$$\langle\mp 00|\hat{\psi}_\pm = 0, \qquad \langle\mp 00|\hat{\psi}_\mp = \langle\pm 00|. \tag{21}$$

We may introduce the notation $\alpha, \beta, \ldots$ for the anticommuting eigenvalues of $\hat{\psi}_\pm$. Consistency requires

$$\alpha\hat{\psi}_\pm = -\hat{\psi}_\pm\alpha, \qquad \alpha|00\pm\rangle = \pm|00\pm\rangle\alpha. \tag{22}$$

The eigenstates of $\hat{q}$, $\hat{\psi}_-$ can be constructed as

$$|q\alpha-\rangle = \mathrm{e}^{-\mathrm{i}q\hat{p}-\alpha\hat{\psi}_+}|00-\rangle, \tag{23}$$

and thus,

$$\hat{q}|q\alpha-\rangle = q|q\alpha-\rangle, \qquad \hat{\psi}_-|q\alpha-\rangle = \alpha|q\alpha-\rangle. \tag{24}$$

Then, the $\hat{\pi}_q$ and $\hat{\psi}_+$ eigenstates are obtained by Fourier transformation:

$$\begin{aligned} |q\beta+\rangle &= -\int \mathrm{d}\alpha\, \mathrm{e}^{\alpha\beta}|q\alpha-\rangle, \qquad |\pi_q\alpha\pm\rangle = -\int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}q\pi_q}|q\alpha\pm\rangle, \\ |p\beta+\rangle &= -\int \mathrm{d}\alpha\, \mathrm{e}^{\alpha\beta}|p\alpha-\rangle, \qquad |\pi_p\alpha\pm\rangle = -\int \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p\pi_p}|p\alpha\pm\rangle, \end{aligned} \tag{25}$$

which gives

$$\begin{aligned} \hat{\pi}_q|\pi_q\alpha\pm\rangle &= \pi_q|\pi_q\alpha\pm\rangle, \qquad \hat{\psi}_+|(q,\pi_q)\beta+\rangle = \beta|(q,\pi_q)\beta+\rangle, \\ \hat{\pi}_p|\pi_p\alpha\pm\rangle &= \pi_p|\pi_p\alpha\pm\rangle, \qquad \hat{\psi}_+|(p,\pi_p)\beta+\rangle = \beta|(p,\pi_p)\beta+\rangle. \end{aligned} \tag{26}$$

This incorporated into the continuity of the spectrum is designed to yield

$$\begin{aligned} \langle\pm\alpha^* q|q'\beta\pm\rangle &= \mathrm{e}^{\alpha\beta}\delta(q-q'), \qquad \langle\mp\alpha^* q|q'\beta\pm\rangle = \mp\delta(\alpha-\beta)\delta(q-q'), \\ \langle\pm\alpha^* p|p'\beta\pm\rangle &= \mathrm{e}^{\alpha\beta}\delta(p-p'), \qquad \langle\mp\alpha^* p|p'\beta\pm\rangle = \mp\delta(\alpha-\beta)\delta(p-p'), \\ \langle\pm\alpha^*\pi_q|q\beta\pm\rangle &= \mathrm{e}^{\alpha\beta}\,\mathrm{e}^{-\mathrm{i}q\pi_q}, \qquad \langle\mp\alpha^*\pi_q|q\beta\pm\rangle = \mp\delta(\alpha-\beta)\,\mathrm{e}^{-\mathrm{i}q\pi_q}, \\ \langle\pm\alpha^*\pi_p|p\beta\pm\rangle &= \mathrm{e}^{\alpha\beta}\,\mathrm{e}^{-\mathrm{i}p\pi_p}, \qquad \langle\mp\alpha^*\pi_p|p\beta\pm\rangle = \mp\delta(\alpha-\beta)\,\mathrm{e}^{-\mathrm{i}p\pi_p}, \end{aligned} \tag{27}$$

where the anticommuting δ function is defined by

$$\int \mathrm{d}\beta\,\delta(\beta-\alpha)\varphi(\beta) = \varphi(\alpha), \qquad \delta(\beta-\alpha) = -\delta(\alpha-\beta). \tag{28}$$

The following completeness relations hold:

$$
\begin{aligned}
&-\int \mathrm{d}\alpha\, \mathrm{d}q |q\alpha\pm\rangle\langle\mp\alpha^* q| = 1, \qquad &&-\int \mathrm{d}\alpha \frac{\mathrm{d}\pi_q}{2\pi} |\pi_q\alpha\pm\rangle\langle\mp\alpha^*\pi_q| = 1,\\
&-\int \mathrm{d}\alpha\, \mathrm{d}p |p\alpha\pm\rangle\langle\mp\alpha^* p| = 1, \qquad &&-\int \mathrm{d}\alpha \frac{\mathrm{d}\pi_p}{2\pi} |\pi_p\alpha\pm\rangle\langle\mp\alpha^*\pi_p| = 1.
\end{aligned}
\tag{29}
$$

The operators equations (13) and (14) allow us consequently to define a pair of appropriate nilpotent supercharges (for such a particle, with unit mass)

$$
\begin{aligned}
&Q_{q+} = B_{q+}\otimes\hat{\psi}_+ = \begin{pmatrix} 0 & B_{q+}\\ 0 & 0\end{pmatrix}, \qquad &&Q_{q-} = B_{q-}\otimes\hat{\psi}_- = \begin{pmatrix} 0 & 0\\ B_{q-} & 0\end{pmatrix},\\
&Q_{p+} = B_{p+}\otimes\hat{\psi}_+ = \begin{pmatrix} 0 & B_{p+}\\ 0 & 0\end{pmatrix}, \qquad &&Q_{p-} = B_{p-}\otimes\hat{\psi}_- = \begin{pmatrix} 0 & 0\\ B_{p-} & 0\end{pmatrix},
\end{aligned}
\tag{30}
$$

which obey the required relations $\{Q_{q\pm}, Q_{q\pm}\} = 0 = \{Q_{p\pm}, Q_{p\pm}\}$. The operators $B_{q\pm}$ can be presented as $B_{q\pm} = B_{q1} \pm \mathrm{i}B_{q2}$, where B_{q1} and B_{q2} are the Hermitian operators. Accordingly, the operators Q_{q1} and Q_{q2} read

$$
\begin{aligned}
Q_{q1} &= Q_{q+} + Q_{q-} = B_{q1}\sigma_1 - B_{q2}\sigma_2,\\
Q_{q2} &= -\mathrm{i}(Q_{q+} - Q_{q-}) = B_{q1}\sigma_2 + B_{q2}\sigma_1,
\end{aligned}
\tag{31}
$$

and similar relations hold for the operators Q_{p1} and Q_{p2}. It is easily verified that $Q_{q\pm}$ are the generators of SUSY transformations between q and $\hat{\psi}$, as well as $Q_{p\pm}$ are the generators of SUSY transformations between p and $\hat{\psi}$:

$$
\begin{aligned}
&[Q_{q\pm}, q] = -\mathrm{i}\hat{\psi}_\pm, \qquad &&[Q_{q\pm}, \pi_q] = \mp W'_q(q)\hat{\psi}_\pm,\\
&\{Q_{q\pm}, \hat{\psi}_\mp\} = p + \pi_q \pm i W(q), \qquad &&\{Q_{q\mp}, \hat{\psi}_\mp\} = 0,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&[Q_{p\pm}, p] = -\mathrm{i}\hat{\psi}_\pm, \qquad &&[Q_{p\pm}, \pi_p] = \mp V'_p(p)\hat{\psi}_\pm,\\
&\{Q_{p\pm}, \hat{\psi}_\mp\} = q + \pi_p \pm \mathrm{i} V(p), \qquad &&\{Q_{p\mp}, \hat{\psi}_\mp\} = 0.
\end{aligned}
\tag{33}
$$

In accordance with equation (31), the SUSY Hamiltonians in q and p representations read

$$
\begin{aligned}
2H_q &= Q_{q1}^2 = Q_{q2}^2 = \{Q_{q+}, Q_{q-}\} = \{B_{q+}, B_{q-}\} + [B_{q+}, B_{q-}]\sigma_3,\\
2H_p &= Q_{p1}^2 = Q_{p2}^2 = \{Q_{p+}, Q_{p-}\} = \{B_{p+}, B_{p-}\} + [B_{p+}, B_{p-}]\sigma_3.
\end{aligned}
\tag{34}
$$

Along the trajectories in (p, q) space, however, this produces the extended Hamiltonian H_{ext}:

$$
H_{\mathrm{ext}} = \begin{pmatrix} H_+ & 0\\ 0 & H_-\end{pmatrix} = H_q - H_p = \begin{pmatrix} H_{q+} - H_{p+} & 0\\ 0 & H_{q-} - H_{p-}\end{pmatrix},
\tag{35}
$$

where

$$
\begin{aligned}
H_q &= \begin{pmatrix} H_{q+} & 0\\ 0 & H_{q-}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{q+}B_{q-} & 0\\ 0 & B_{q-}B_{q+}\end{pmatrix},\\
H_p &= \begin{pmatrix} H_{p+} & 0\\ 0 & H_{p-}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{p+}B_{p-} & 0\\ 0 & B_{p-}B_{p+}\end{pmatrix}.
\end{aligned}
\tag{36}
$$

Equations (35) and (36) incorporating with equation (13) yield

$$
\begin{aligned}
H_{\mathrm{ext}} = \tfrac{1}{2}[&(p + \pi_q)^2 - (q + \pi_p)^2 + W^2(q) - V^2(p)\\
&- \mathrm{i}\sigma_3(p + \pi_q)W(q) + \mathrm{i}\sigma_3(q + \pi_p)V(p)].
\end{aligned}
\tag{37}
$$

Let $\chi_r(q, p)$ be an eigenfunction of H_{ext} corresponding to the eigenvalue E_r,

$$H_{\mathrm{ext}}\chi_r(q, p) = E_r\chi_r(q, p), \tag{38}$$

where by virtue of equation (6), the $\chi_r(q, p)$ (for combined index $r \equiv (\alpha, \beta)$) is in the form

$$\chi_r(q, p) = \overline{\chi}_r(q, p)\,\mathrm{e}^{-\mathrm{i}pq} \equiv \chi_{(\alpha,\beta)}(q, p) = \psi_\alpha(q)\phi^*_\beta(p)\,\mathrm{e}^{-\mathrm{i}pq}. \tag{39}$$

The exponential factor is a consequence of the total time derivative, $-\mathrm{d}(qp)/\mathrm{d}t$, in equation (1). It is easily verified that

$$(p + \pi_q)\chi_r(q, p) = (\pi_q\overline{\chi}_r(q, p))\,\mathrm{e}^{-\mathrm{i}pq}, \tag{40}$$

and so on. Substitution of equations (39) and (40) in equation (38) gives the stationary states of the system which are the normalizable solutions of the Schrödinger equation. With this observation we arrive at

$$\overline{H}_{\mathrm{ext}}\overline{\chi}_r(q, p) = E_r\overline{\chi}_r(q, p), \tag{41}$$

provided by the reduced Hamiltonian, $\overline{H}_{\mathrm{ext}}$,

$$\overline{H}_{\mathrm{ext}} = \begin{pmatrix} \overline{H}_+ & 0 \\ 0 & \overline{H}_- \end{pmatrix} = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + W^2(q) - V^2(p) - \sigma_3(W'_q(q) - V'_p(p))\right]. \tag{42}$$

Along the actual trajectories in q space, equation (42) reproduces the results obtained in [7]. A prime will indicate the differentiation with respect either to q or p spaces. From now on, we replace H_{ext} by $\overline{H}_{\mathrm{ext}}$ and $\chi_r(q, p)$ by $\overline{\chi}_r(q, p)$, respectively, and retain former notational conventions. This realization characterizes a non-interacting point particle of mass $m = 1$ moving along the real line in (q, p) space under influence of the external scalar potential

$$U_\pm = U_{q\pm} - U_{p\pm} \equiv W^2(q) - V^2(p) \mp (W'_q(q) - V'_p(p)). \tag{43}$$

The time evolution of the state $|t\rangle$ is given

$$\chi_-(q\alpha p\beta t) = -\int \mathrm{d}\alpha'\,\mathrm{d}q'\,\mathrm{d}\beta'\,\mathrm{d}p'\,K(q\alpha p\beta t|q'\alpha' p'\beta' t'), \tag{44}$$

provided by the kernel

$$K(q\alpha p\beta t|q'\alpha' p'\beta' t') = \langle +q\alpha^* p\beta^*|\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{ext}}(t-t')}|q'\alpha' p'\beta'\rangle, \tag{45}$$

which can be evaluated by the path integral. Actually, an alternative approach to describe the state space and dynamics of the extended phase-space quantum system is by the path integral [2], which reads

$$\mathcal{K}_{ff'}(qpt|q'p't') = \langle qpf|\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{ext}}(t-t')}|q'p'f'\rangle, \tag{46}$$

where the extended SUSY Hamiltonian, given by equation (42), can be represented as

$$H_{\mathrm{ext}} = \tfrac{1}{2}\left(\pi_q^2 + W^2(q) + iW'_q(q)[\hat{\psi}_1, \hat{\psi}_2]\right) - \tfrac{1}{2}\left(\pi_p^2 + V^2(p) + \mathrm{i}V'_q(p)[\hat{\psi}_1, \hat{\psi}_2]\right). \tag{47}$$

To infer the extended Hamiltonian equation (47) equivalently one may start from the c-number extended Lagrangian of extended phase-space quantum field theory in (0 + 1) dimensions in q and p spaces:

$$\begin{aligned}\mathcal{L}_{\mathrm{ext}}(p, q, \dot{p}, \dot{q}) = -\dot{q}p - q\dot{p} + \frac{1}{2}\left[\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - W^2(q) + \psi^T\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + W'_q(q)\sigma_2\right)\psi\right] \\ + \frac{1}{2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 - V^2(p) + \psi^T\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + V'_p(p)\sigma_2\right)\psi\right],\end{aligned} \tag{48}$$

where $\psi = \binom{\psi_1}{\psi_2}$, the components of which are to be interpreted as anticommuting c-numbers. Equation (48) can be re-written as

$$\mathcal{L}_{\text{ext}}(p, q, \dot{p}, \dot{q}) = -\dot{q}p - q\dot{p} + \frac{1}{2}\left[\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - W^2(q)\right] + fW'_q(q) + \frac{1}{2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 - V^2(p)\right] + fV'_p(p). \tag{49}$$

With the Hamiltonian H_{ext}, the path integral equation (46) is diagonal:

$$\mathcal{K}_{ff'}(qpt|q'p't') = \mathcal{K}_{ff'}(qt|q't')\mathcal{K}_{ff'}(pt|p't') = \delta_{ff'}\int_{q'}^{q}\mathcal{D}q\int_{p'}^{p}\mathcal{D}p\exp\left(\mathrm{i}\int_{t'}^{t}\mathcal{L}_{\text{ext}}(p, q, \dot{p}, \dot{q})\,\mathrm{d}t\right). \tag{50}$$

Knowing the path integral equation (50), it is sufficient to specify the initial wavefunction $\chi_f(q', p', t')$ to obtain all possible information about the system at any later time t, by

$$\chi_f(q, p, t) = \sum_{f'}\int \mathrm{d}q'\mathrm{d}p'\mathcal{K}_{ff'}(qpt|q'p't')\chi_{f'}(q', p', t'), \tag{51}$$

with $\chi_{\pm 1/2}(q, p, t) = \chi_{1,2}(q, p, t)$ (equation (17)).

4. The vacuum energy and the topology of superpotential

The supersymmetry equation (7) of quantum system is said to be a good symmetry (good SUSY) if the ground-state energy of H_{ext} vanishes. In the other case, $\inf\operatorname{spec} H_{\text{ext}} > 0$, SUSY is said to be broken. Note that under the replacement of SUSY potentials, $W \to -W$ and $V \to -V$, $(U_\pm \to -U_\mp)$, the roles of the two Hamiltonians H_+ and H_- are interchanged. Hence, the sign of the SUSY potentials may be fixed by some convention. For good SUSY the ground state χ_0 of H_{ext} either belongs to H_+ or H_-

$$H_\pm\chi_0^\pm = 0 \Leftrightarrow B_\mp\chi_0^\pm = 0. \tag{52}$$

As far as $H_{q\pm}$ and $H_{p\pm}$ are independent, equation (52) gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}q} \pm W(q)\right)\psi_0^\pm(q) = 0, \qquad \left(\frac{\mathrm{d}}{\mathrm{d}p} \pm V(p)\right)\phi_0^\pm(p) = 0, \tag{53}$$

provided with $\chi_0^\pm(q, p)\rangle \equiv \psi_0^\pm(q)\phi_0^\pm(p)$, and

$$\langle q|\psi_0^\pm\rangle \equiv \psi_0^\pm(q), \qquad \langle p|\phi_0^\pm\rangle \equiv \phi_0^\pm(p). \tag{54}$$

The functions $\psi_0^\pm(q)$ and $\phi_0^\pm(p)$ have to be square-integrable for SUSY to be a good symmetry. This requirement puts conditions on the SUSY potentials:

$$\begin{aligned} &\int_0^\infty W(q')\,\mathrm{d}q' \to \infty \qquad \text{at} \quad q \to \pm\infty \qquad \text{for} \quad \psi_0^+,\\ &\int_0^\infty W(q')\,\mathrm{d}q' \to -\infty \qquad \text{at} \quad q \to \pm\infty \qquad \text{for} \quad \psi_0^-, \end{aligned} \tag{55}$$

and similar ones hold for $V(p)$. Depending on the asymptotic behavior of the SUSY potentials one of the two functions $\chi_0^\pm$ will be normalizable (good SUSY) or both are not normalizable (broken SUSY). For continuous SUSY potentials $U_\pm(p, q)$ the functions $W(q)$ and $V(p)$ must have an odd number of zeros (counted with their multiplicity) for SUSY to be good. A continuous SUSY potentials with an even number of zeros necessarily leads to a broken

SUSY as the functions equation (55) will be not square-integrable. Consequently, if $W(q)$ and $V(p)$ have a well-defined parity, and odd $W(q)$ and $V(p)$ lead to good SUSY, whereas an even $W(q)$ and $V(p)$ break SUSY:

$$\begin{aligned} &W(-q) = -W(q) \Rightarrow U_{q\pm}(-q) = U_{q\pm}(q) \quad \text{(SUSY and parity are} \\ &\qquad\qquad \text{good in } q \text{ subspace)}, \\ &W(-q) = W(q) \Rightarrow U_{q\pm}(-q) \neq U_{q\pm}(q) \quad \text{(SUSY and parity are} \\ &\qquad\qquad \text{broken in } q \text{ subspace)}, \end{aligned} \tag{56}$$

and correspondingly the similar conditions hold for $V(p)$ and $U_{q\pm}(p)$. The spectra of H_+ and H_- are related as follows:

$$\begin{aligned} &\mathrm{spec}(H_-)/\{0\} = \mathrm{spec}(H_+) \qquad \text{(good SUSY)}, \\ &\mathrm{spec}(H_-) = \mathrm{spec}(H_+) \qquad \text{(broken SUSY)}. \end{aligned} \tag{57}$$

Clearing up this situation, the Witten index is turned out to be one of the useful tools which, according to the Atiyah–Singer index theorem [11] associates with the operator index and depends only on the asymptotic values of SUSY potentials. This is a topological characteristic and does not vary with the variation of the parameters of theory. Thus, the Witten index reads

$$\Delta(\beta) = tr(P\,\mathrm{e}^{-\beta H_{\mathrm{ext}}}) = \mathrm{tr}(P\,\mathrm{e}^{-\beta(H_q - H_p)}), \qquad \beta > 0. \tag{58}$$

For a pure point spectrum of H_{ext} this index is the difference of the number of spin-up states ($\uparrow$) and spin-down states ($\downarrow$) with zero energy:

$$\Delta(\beta) = N_{\uparrow}(E=0) - N_{\downarrow}(E=0). \tag{59}$$

Note that the factor $\mathrm{e}^{-\beta H_{\mathrm{ext}}}$ in equation (58) has only been introduced for the regularization of the trace. The conditions of the positive-energy eigenstates cancel due to the pairwise degeneracy mentioned above. For a continuous spectrum this is not the case as the spectral densities for the spin-up and spin-down states are in general different due to which Witten index becomes β dependent [12]. Therefore, for simplicity, we assume purely discrete spectra. Then, equation (58) yields

$$\begin{aligned} \Delta &= \mathrm{ind}B = \dim\ker H_- - \dim\ker H_+ \\ &= \dim\ker H_{q-} + \dim\ker H_{p+} - \dim\ker H_{q+} - \dim\ker H_{p-}. \end{aligned} \tag{60}$$

Introducing in equation (60) a set of H_q and H_p upon reduction to the q or p spaces makes provisions for equation (59), which incorporated into equation (55) are designed to yield

$$\Delta = \tfrac{1}{2}[\mathrm{sgn}W(+\infty) + \mathrm{sgn}V(-\infty) - \mathrm{sgn}W(-\infty) - \mathrm{sgn}V(+\infty)]. \tag{61}$$

Hence for good SUSY, one has $\Delta = \pm 1$ with the ground state belonging to $H_\pm$. For broken SUSY, one has $\Delta = 0$. The generalized Witten index, Δ, shows new features of the present formulation over the well-known ordinary SUSY QM formalism. Actually, even the SUSY was broken in q or p spaces, it can be still good in (q, p) space, or the SUSY was good in q or p spaces, it can be broken in (q, p) space.

The spectral properties of $H_\pm$ are summarized in the following table:

$$\begin{aligned} &\Delta = +1: \qquad E_n^+ = E_{n+1}^- > 0, \qquad E_0^- = 0, \\ &\Delta = -1: \qquad E_n^- = E_{n+1}^+ > 0, \qquad E_0^+ = 0, \\ &\Delta = 0: \qquad E_n^+ = E_n^- > 0, \end{aligned} \tag{62}$$

where $E_n^\pm, n = 0, 1, 2 \ldots$, denotes the ordered set of eigenvalues of $H_\pm$ with $E_n^\pm < E_{n+1}^\pm$, which, in turn, are dependent on spectra of $H_{q\pm}$ and $H_{p\pm}$.

5. A shape invariance of exactly solvable SUSY potentials

An extended Hamiltonian H_{ext} can be treated as a set of two ordinary two-dimensional partner Hamiltonians

$$H_\pm = \tfrac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_\pm(q, p)\right]. \tag{63}$$

Due to SUSY they have the same energy spectra at arbitrary functions $W(q)$ and $V(p)$, except the ground state of H_- (defined in accordance with usual convention) which has no corresponding state in the spectra of H_+.

In [10], it was shown that a subset of the SUSY potentials for which the Schrödinger-like equations are exactly solvable share an integrability conditions called *shape invariance*. The partner potentials $U_\pm(q, p)$ equation (43) are called shape invariant if they satisfy an integrability condition

$$U_+(a, q, p) = U_-(a_1, q, p) + R(a), \qquad a_1 = f(a), \tag{64}$$

where a and a_1 are a set of parameters that specify phase-space-independent properties of the potentials, and the reminder $R(a)$ is independent of (q, p). Although this looks like a satisfactory state of affairs, we may not always be so fortunate to have such potentials at our disposal. In fact a shape invariance is not the most general integrability condition as not all exactly solvable potentials seem to be shape invariant [12].

Using the standard technique, we construct a series of Hamiltonians H_n, $n = 0, 1, 2, \ldots,$

$$H_n = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_-(a_n, q, p) + \sum_{k=1}^{n} R(a_k)\right], \tag{65}$$

where $a_n = f^{(n)}(a)$ (n is the number of iterations). Comparing the spectra H_n and H_{n+1}, due to equation (64), we obtain

$$H_{n+1} = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_+(a_n, q, p) + \sum_{k=1}^{n} R(a_k)\right]. \tag{66}$$

We see that the Hamiltonians H_n and H_{n+1} have the same energy spectra, except the ground state of H_n, the energy of which as can be found from equation (65) is equal to $\sum_{k=1}^{n} R(a_k)$. Going through H_n to H_{n-1} and so on, we subsequently obtain the initial Hamiltonian $H_0 = H_- = \frac{1}{2}\left[\pi_q^2 - \pi_p^2 + U_-(a, q, p)\right]$, the ground state of which is equal zero, but all the other energy levels coincide with the lower levels of Hamiltonians H_n. Continuing along this line, the entire energy spectrum of H_{ext} is $\widetilde{E}_n = \sum_{k=1}^{n} R(a_k)$. Hence the spectrum of Hamiltonian with the potential $U(a, q, p) = U_-(a, q, p) + C(a)$ has the form

$$E_n = \widetilde{E}_n + C(a) = \sum_{k=1}^{n} R(a_k) + C(a). \tag{67}$$

Instead of developing the full machinery here, we will illustrate this in passing in the following example.

Example of Scarf potential. To demonstrate practical merits of the concept of shape invariance, we now obtain analytic expressions for the entire energy spectrum in case of *Scarf* potential,

$$U(a, b, q, p) = U_q(a, b) - U_p(b, p) = -\frac{a(a+1)}{2ch^2 q} + \frac{b(b+1)}{2ch^2 p}, \tag{68}$$

without ever referring to an underlying differential equation. In case at hand we have $W(q) = a\, th\, q$ and $V(p) = b\, th\, p$, hence

$$U_\pm(a, b, q, p) = U_{q\pm}(a, q) - U_{p\pm}(b, p) = -\frac{a(a \mp 1)}{2ch^2 q} + \frac{a^2}{2} + \frac{b(b \pm 1)}{2ch^2 p} - \frac{b^2}{2}. \tag{69}$$

This yields

$$
\begin{aligned}
&a_1 = f_1(a) = a - 1; \qquad && a_n = a - n; \qquad && C_1(a) = -\frac{a^2}{2}, \\
&b_1 = f_2(b) = b - 1; \qquad && b_m = b - m; \qquad && C_2(b) = -\frac{b^2}{2}, \\
&\sum_{k=1}^{n} R_1(a_k) = \frac{a^2 - a_n^2}{2}, \qquad && \sum_{k=1}^{m} R_2(b_k) = \frac{b^2 - b_m^2}{2}.
\end{aligned}
\tag{70}
$$

The entire energy spectrum of the H_{ext} can be easily obtained as

$$
E_{nm} = E_{qn} - E_{pm} = -\frac{a_n^2}{2} + \frac{b_m^2}{2} = -\frac{(a-n)^2}{2} + \frac{(b-m)^2}{2}. \tag{71}
$$

The shape invariance has an underlying algebraic structure of Lie algebras [10], which transform the parameters of the potentials. Shape-invariance algebra in general is an infinite dimensional. However, under some conditions they become finite dimensional. The Hamiltonian of exactly solvable systems can be written as a linear or quadratic function of an underlying algebra, and all the quantum states of these systems can be determined by the independent group theoretical methods with a general change of parameters which involves nonlinear extensions of Lie algebras [13]. A more detailed analysis and calculations on the independent group theoretical methods with nonlinear extensions of Lie algebras in context of extended phase-space formulation of quantum mechanics will be presented in another paper to follow at a later date.

6. Conclusions

The theory we considered can be regarded as a quantum field theory in $(0 + 1)$ dimensions in q and p spaces, which exhibits supersymmetry. We construct the $(N = 2)$ realization of the extended phase-space SUSY algebra and discuss the vacuum energy and topology of super-potentials. The question of spontaneously breaking of extended SUSY deserves further investigation.

We demonstrate the merits of shape invariance of exactly solvable extended SUSY potentials, which has underlying algebraic structure, by obtaining analytic expressions for the entire energy spectrum of extended Hamiltonian with Scarf potential without ever referring to underlying differential equation.

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